Access to Science, Engineering and Agriculture: Mathematics 1 MATH00030 Chapter 8 Solutions

1. (a) (i) The mean is $\overline{x} = \dfrac{1}{7}(-3 + (-9) + 6 + 2 + 1 + 2 + 6) = \dfrac{5}{7}$.

(ii) Since we are going to calculate the interquartile range below, we will put the whole list into ascending order, although this is not necessary to find the median. The list in ascending order is $-9, -3, 1, 2, 2, 6, 6$. Since there are seven numbers (an odd number), the median is $m = x_{\frac{7+1}{2}} = x_4 = 2$.

(iii) There are 2 twos, 2 sixes and one of each of the other numbers, so there are two modes, 2 and 6.

(iv) The variance is (where I have used the original ordering)

$$\begin{aligned}
\text{Var}(x) =& \frac{\sum_{i=1}^{7}(x_i - \overline{x})^2}{7} \\
=& \frac{\left(-3 - \frac{5}{7}\right)^2 + \left(-9 - \frac{5}{7}\right)^2 + \left(6 - \frac{5}{7}\right)^2 + \left(2 - \frac{5}{7}\right)^2}{7} \\
&+ \frac{\left(1 - \frac{5}{7}\right)^2 + \left(2 - \frac{5}{7}\right)^2 + \left(6 - \frac{5}{7}\right)^2}{7} \\
=& \frac{1172/7}{7} \\
=& \frac{1172}{49} \\
\simeq& 23.918.
\end{aligned}$$

(v) The standard deviation is $\sigma = \sqrt{\text{Var}(x)} = \sqrt{\dfrac{1172}{49}} \simeq 4.891$.

(vi) Since we have an odd number of numbers, we discard the median and split the remaining numbers into a lower half $-9, -3, 1$ and an upper half $2, 6, 6$. There are three numbers in each of these new groups (an odd number), so in each case the median is $x_{\frac{3+1}{2}} = x_2$. Thus the lower quartile is $Q_1 = -3$ and the upper quartile is $Q_3 = 6$. Hence the interquartile range is $Q_3 - Q_1 = 6 - (-3) = 9$.

(b) (i) The mean is

$$\begin{aligned}
\overline{x} &= \frac{1}{8}(-3 + (-4) + (-9) + (-2) + (-5) + (-6) + (-6) + (-7)) \\
&= \frac{-42}{8} \\
&= -\frac{21}{4}.
\end{aligned}$$

(ii) The list in ascending order is $-9, -7, -6, -6, -5, -4, -3, -2$.
Since there are eight numbers (an even number), the median is
$m = \frac{x_{\frac{8}{2}} + x_{\frac{8}{2}+1}}{2} = \frac{x_4 + x_5}{2} = \frac{-6 + (-5)}{2} = -\frac{11}{2}$.

(iii) There are two minus sixes and one of each of the other numbers, the mode is -6.

(iv) The variance is (where I have used the original ordering)

$$
\begin{aligned}
\mathrm{Var}(x) =& \frac{\sum_{i=1}^{8}(x_i - \overline{x})^2}{8} \\
=& \frac{\left(-3-\left(-\frac{21}{4}\right)\right)^2 + \left(-4-\left(-\frac{21}{4}\right)\right)^2 + \left(-9-\left(-\frac{21}{4}\right)\right)^2}{8} \\
&+ \frac{\left(-2-\left(-\frac{21}{4}\right)\right)^2 + \left(-5-\left(-\frac{21}{4}\right)\right)^2 + \left(-6-\left(-\frac{21}{4}\right)\right)^2}{8} \\
&+ \frac{\left(-6-\left(-\frac{21}{4}\right)\right)^2 + \left(-7-\left(-\frac{21}{4}\right)\right)^2}{8} \\
=& \frac{71/2}{8} \\
=& \frac{71}{16} \\
=& 4.4375.
\end{aligned}
$$

(v) The standard deviation is $\sigma = \sqrt{\mathrm{Var}(x)} = \sqrt{\frac{71}{16}} \simeq 2.107$.

(vi) Since there are eight numbers (an even number) we just split the numbers into a lower half $-9, -7, -6, -6$ and an upper half $-5, -4, -3, -2$.
There are four numbers in each of these new groups (an even number), so the median is $\frac{x_{\frac{4}{2}} + x_{\frac{4}{2}+1}}{2} = \frac{x_2 + x_3}{2}$.
Hence the lower quartile is $Q_1 = \frac{-7 + (-6)}{2} = -\frac{13}{2}$
and the upper quartile is $Q_3 = \frac{-4 + (-3)}{2} = -\frac{7}{2}$.
Thus the interquartile range is $Q_3 - Q_1 = -\frac{7}{2} - \left(-\frac{13}{2}\right) = 3$.
Note the interquartile range always has to be greater than or equal to zero, even if the numbers are all negative.

(c) (i) The mean is $\overline{x} = \frac{1}{9}(3 + 3 + 4 + 2 + 1 + 5 + 2 + 10 + 0) = \frac{30}{9}$.

(ii) The list in ascending order is $0, 1, 2, 2, 3, 3, 4, 5, 10$. Since there are nine numbers (an odd number), the median is
$m = x_{\frac{9+1}{2}} = x_5 = 3$.

(iii) There are 2 twos, 2 threes and one of each of the other numbers, so there are two modes, 2 and 3.

(iv) The variance is (where I have used the original ordering)

$$\begin{aligned}
\text{Var}(x) =& \frac{\sum_{i=1}^{9}(x_i - \overline{x})^2}{9} \\
=& \frac{\left(3-\frac{30}{9}\right)^2 + \left(3-\frac{30}{9}\right)^2 + \left(4-\frac{30}{9}\right)^2 + \left(2-\frac{30}{9}\right)^2 + \left(1-\frac{30}{9}\right)^2}{9} \\
&+ \frac{\left(5-\frac{30}{9}\right)^2 + \left(2-\frac{30}{9}\right)^2 + \left(10-\frac{30}{9}\right)^2 + \left(0-\frac{30}{9}\right)^2}{9} \\
=& \frac{68}{9} \\
\simeq& 7.556.
\end{aligned}$$

(v) The standard deviation is $\sigma = \sqrt{\text{Var}(x)} = \sqrt{\frac{68}{9}} \simeq 2.749$.

(vi) Since we have an odd number of numbers, we discard the median and split the remaining numbers into a lower half $0, 1, 2, 2$ and an upper half $3, 4, 5, 10$. There are four numbers in each of these new groups (an even number), so in each case the median is $\frac{x_{\frac{4}{2}} + x_{\frac{4}{2}+1}}{2} = \frac{x_2 + x_3}{2}$. Thus the lower quartile is $Q_1 = \frac{1+2}{2} = \frac{3}{2}$ and the upper quartile is $Q_3 = \frac{4+5}{2} = \frac{9}{2}$. Hence the interquartile range is $Q_3 - Q_1 = \frac{9}{2} - \frac{3}{2} = 3$.

(d) (i) The mean is

$$\begin{aligned}
\overline{x} &= \frac{1}{10}(-7 + (-2) + (-4) + (-6) + 0 + 4 + 2 + (-5) + (-3) + 7) \\
&= \frac{-14}{10} \\
&= -\frac{7}{5}.
\end{aligned}$$

(ii) The list in ascending order is $-7, -6, -5, -4, -3, -2, 0, 2, 4, 7$.
Since there are ten numbers (an even number), the median is
$m = \frac{x_{\frac{10}{2}} + x_{\frac{10}{2}+1}}{2} = \frac{x_5 + x_6}{2} = \frac{-3 + (-2)}{2} = -\frac{5}{2}$.

(iii) There is one of each number, so all ten numbers in the list are modes.

(iv) The variance is (where I have used the original ordering)

$$
\begin{aligned}
\text{Var}(x) =& \frac{\sum_{i=1}^{10}(x_i - \overline{x})^2}{10} \\
=& \frac{\left(-7-\left(-\frac{7}{5}\right)\right)^2 + \left(-2-\left(-\frac{7}{5}\right)\right)^2 + \left(-4-\left(-\frac{7}{5}\right)\right)^2}{10} \\
&+ \frac{\left(-6-\left(-\frac{7}{5}\right)\right)^2 + \left(0-\left(-\frac{7}{5}\right)\right)^2 + \left(4-\left(-\frac{7}{5}\right)\right)^2}{10} \\
&+ \frac{\left(2-\left(-\frac{7}{5}\right)\right)^2 + \left(-5-\left(-\frac{7}{5}\right)\right)^2 + \left(-3-\left(-\frac{7}{5}\right)\right)^2}{10} \\
&+ \frac{\left(7-\left(-\frac{7}{5}\right)\right)^2}{10} \\
=& \frac{4710/25}{10} \\
=& \frac{471}{25} \\
=& 18.84.
\end{aligned}
$$

(v) The standard deviation is $\sigma = \sqrt{\text{Var}(x)} = \sqrt{\frac{471}{25}} \simeq 4.341.$

(vi) Since there are ten numbers (an even number) we just split the numbers into a lower half $-7, -6, -5, -4, -3$ and an upper half $-2, 0, 2, 4, 7$. There are five numbers in each of these new groups (an odd number), so in each case the median is $x_{\frac{5+1}{2}} = x_3$. Hence the lower quartile is $Q_1 = -5$ and the upper quartile is $Q_3 = 2$. Thus the interquartile range is $Q_3 - Q_1 = 2 - (-5) = 7$.

2. (a) There are nine points, so $n = 9$ and

$$
\sum_{i=1}^{n} x_i = \sum_{i=1}^{9} x_i = -2 + 0 + 2 + 4 + 6 + 8 + 10 + 12 + 14 = 54
$$
$$
\sum_{i=1}^{n} y_i = \sum_{i=1}^{9} y_i = -1 + 0 + 1 + 3 + 3 + 4 + 4 + 5 + 5 = 24
$$

$$
\begin{aligned}
\sum_{i=1}^{n} x_i y_i =& \sum_{i=1}^{9} x_i y_i \\
=& (-2)(-1) + (0)(0) + (2)(1) + (4)(3) + (6)(3) \\
&+ (8)(4) + (10)(4) + (12)(5) + (14)(5) \\
=& 2 + 0 + 2 + 12 + 18 + 32 + 40 + 60 + 70 \\
=& 236.
\end{aligned}
$$

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{9} x_i^2$$
$$=(-2)^2 + 0^2 + 2^2 + 4^2 + 6^2 + 8^2 + 10^2 + 12^2 + 14^2$$
$$=4 + 0 + 4 + 16 + 36 + 64 + 100 + 144 + 196$$
$$=564.$$

Hence

$$m = \frac{n\left(\sum_{i=1}^{n} x_i y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2}$$
$$= \frac{9(236) - (54)(24)}{9(564) - 54^2}$$
$$= \frac{828}{2160}$$
$$= \frac{23}{60}$$
$$\simeq 0.383,$$

and

$$c = \overline{y} - m\overline{x} = \frac{\sum_{i=1}^{9} y_i}{9} - m\frac{\sum_{i=1}^{9} x_i}{9} = \frac{24}{9} - \frac{23}{60} \times \frac{54}{9} = \frac{11}{30} \simeq 0.367.$$

Thus the line of best fit is $y = \frac{23}{60}x + \frac{11}{30}$.
The points and the graph are shown in Figure 1.

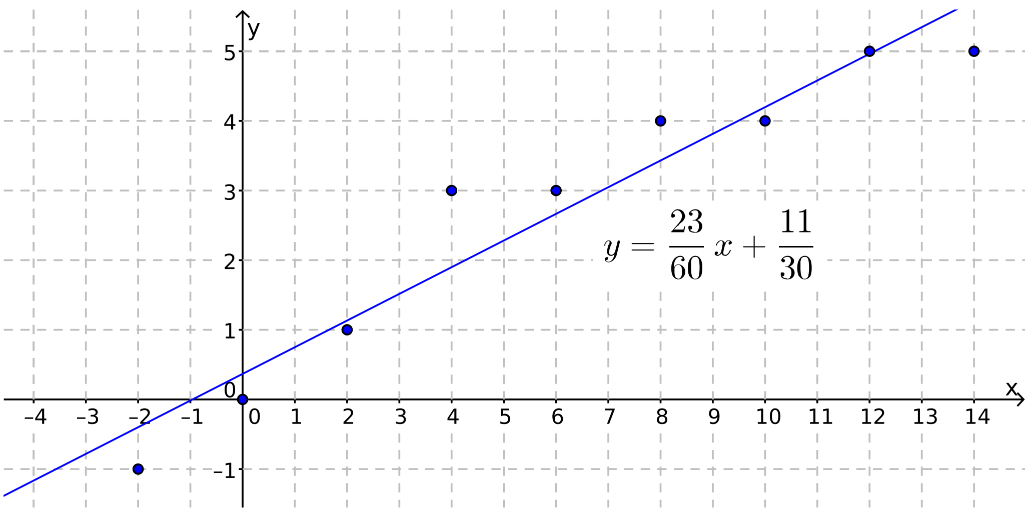

Figure 1: The Line of Best Fit and Points From Exercise 2(a).

(b) There are eleven points, so $n = 11$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{11} x_i = -3 + (-2) + 0 + 1 + 2 + 5 + 6 + 7 + 9 + 12 + 13 = 50$$
$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{11} y_i = 3 + 4 + 3 + 2 + 3 + 3 + 1 + 2 + 1 + 1 + 0 = 23$$

$$\begin{aligned}\sum_{i=1}^{n} x_i y_i = &\sum_{i=1}^{11} x_i y_i \\ =&(-3)(3) + (-2)(4) + (0)(3) + (1)(2) + (2)(3) + (5)(3) \\ &+ (6)(1) + (7)(2) + (9)(1) + (12)(1) + (13)(0) \\ =& -9 + (-8) + 0 + 2 + 6 + 15 + 6 + 14 + 9 + 12 + 0 \\ =&47.\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n} x_i^2 = &\sum_{i=1}^{11} x_i^2 \\ =&(-3)^2 + (-2)^2 + 0^2 + 1^2 + 2^2 + 5^2 + 6^2 + 7^2 + 9^2 + 12^2 + 13^2 \\ =&9 + 4 + 0 + 1 + 4 + 25 + 36 + 49 + 81 + 144 + 169 \\ =&522.\end{aligned}$$

Hence

$$\begin{aligned} m &= \frac{n\left(\sum_{i=1}^{n} x_i y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2} \\ &= \frac{11(47) - (50)(23)}{11(522) - 50^2} \\ &= \frac{-633}{3242} \\ &= -\frac{633}{3242} \\ &\simeq -0.195,\end{aligned}$$

and

$$c = \overline{y} - m\overline{x} = \frac{\sum_{i=1}^{11} y_i}{11} - m\frac{\sum_{i=1}^{11} x_i}{11} = \frac{23}{11} - \left(-\frac{633}{3242}\right) \times \frac{50}{11} = \frac{4828}{1621} \simeq 2.978.$$

Thus the line of best fit is $y = -\dfrac{633}{3242}x + \dfrac{4828}{1621}$.
The points and the graph are shown in Figure 2.

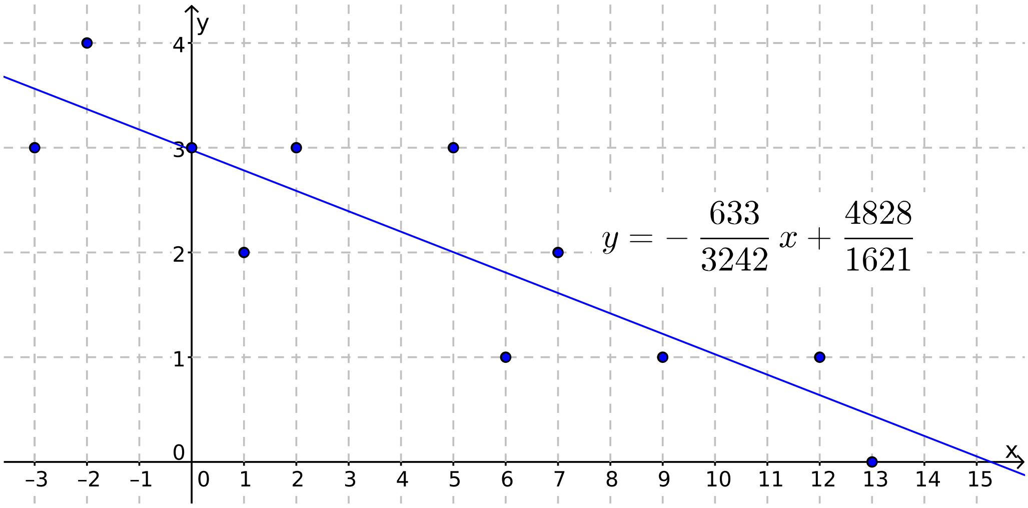

Figure 2: The Line of Best Fit and Points From Exercise 2(b).